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THE NON-LINEAR DEFORMATION AND STABILITY OF ELLIPTICAL CYLINDRICAL SHELLS UNDER TRANSVERSE BENDING†

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A finite-element formulation of the solution of problems of the stability of non-circular cylindrical shells taking into account their bending moments and the non-linearity of their precritical stress-strain state in proposed. Explicit expressions for the displacements of the elements of non-circular cylindrical shells as rigid bodies are derived by integration of the equations obtained by equating the components of the linear strains to zero. These expressions are used to construct shape functions of an effective quadrilateral finite element of the natural curvature. An effective algorithm is developed for investigating the non-linear deformation and stability of the shells. The stability of a cylindrical shell of elliptical cross-section under transverse bending is investigated. The influence of the ellipticity and non-linearity of the deformation on the shell's stability is determined. The results of the analysis are compared with experimental data.

The stability of non-circular shells has been insufficiently investigated, unlike the stability of circular shells. Thousands of publications exist on circular shells, but only a few dozen on non-circular ones. This can be explained by the less extensive use of non-circular shells and by the difficulties encountered in solving problems with variable radii of curvature, which lead to the presence of variable coefficients in the equations of stability. Existing solutions of stability problems are obtained by analytical methods, and, as a rule, using a linear approximation and ignoring the bending moments and the non-linearity of the precritical state, i.e. using the classical formulation.

1. THE DISPLACEMENTS OF FINITE ELEMENTS OF NON-CIRCULAR CYLINDRICAL SHELLS AS RIGID BODIES

For displacements of the elements as rigid bodies, the components of the strain are equal to zero. By equating the linear components of the strains, the changes in curvature and the torsion to zero [1], we obtain the equations

$$\begin{aligned} &\varepsilon_1 = u_x = 0, \quad \varepsilon_2 = k_2(\upsilon_\beta + w) = 0, \quad \varepsilon_3 = \upsilon_x + k_2 u_\beta = 0 \\ &\chi_1 = w_{xx} = 0, \quad \chi_2 = k_2[k_2(\upsilon - w_\beta)]_\beta = 0, \quad \chi_3 = [k_2(\upsilon - w_\beta)]_x = 0 \end{aligned} \tag{1.1}$$

Here u, υ and w are the tangential displacements and deflection, R is the radius, $k_2 = R^{-1}$ is curvature of the cross-section, β is the angle between the normal to the cross section and the axis b of the cross-section, and x is a longitudinal coordinate (Fig. 1). The subscripts x and β denote differentiation with respect to the variables x and β.

Let us integrate Eqs (1.1). From the penultimate equation of (1.1), we have

$$\upsilon = w_\beta + RC_5, C_5 = \text{const} \tag{1.2}$$

From the second equation of (1.1) it follows that

$$w = -\upsilon_\beta \tag{1.3}$$

Taking this equality into account, from (1.2) we obtain the equation

$$\upsilon_{\beta\beta} + \upsilon = RC_5$$

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the solution of which has the form

$$v = C_3c + C_4s + v_n; \quad c = \cos\beta, \quad s = \sin\beta \tag{1.4}$$

A particular solution v_n of the non-homogeneous equation is found by varying the arbitrary constants C_3 and C_4. After calculations we have

$$\begin{aligned} v &= C_3c + C_4s - C_5(\psi_1c + \psi_2s) + C_7c + C_8s \\ w &= C_3s - C_4c - C_5(\psi_1s - \psi_2c) + C_7s - C_8c \\ \psi_1 &= \int Rsd\beta, \quad \psi_2 = -\int Rcd\beta \end{aligned} \tag{1.5}$$

The fourth equation of (1.1) is satisfied by setting

$$C_7 = C_2x, \quad C_8 = -C_1x$$

As a result we obtain

$$\begin{aligned} v &= C_3c + C_4s - C_5(\psi_1c + \psi_2s) + (C_2c - C_1s)x \\ w &= C_3s - C_4c - C_5(\psi_1s - \psi_2c) + (C_2s + C_1c)x \end{aligned} \tag{1.6}$$

From the third equation of (1.1) we find

$$u = C_1\psi_1 + C_2\psi_2 + C_6 \tag{1.7}$$

The functions (1.6) and (1.7) satisfy Eqs (1.1), and, therefore, in the linear approximation, the correspond to the displacements of the elements as rigid bodies.

2. THE FINITE ELEMENT AND AN ALGORITHM FOR SOLVING THE PROBLEM

We will divide the shell by lines of principal curvatures into m parts along the generator and into n parts along the director. Hence, the shell will be represented by a set of $m \times n$ curvilinear rectangular finite elements (Fig. 1). Using a bilinear approximation of the deformational tangential displacements and a bicubical approximation for the deflection, taking expressions (1.6) and (1.7) into account, we will write the following expressions for the total displacements of the points of the finite element

$$\begin{aligned} u &= a_1xy + a_2x + a_3y + a_4 + a_6\psi_2 + a_{20}\psi_1 \\ v &= a_5xy + a_6xc + a_7y + a_8(\psi_1c + \psi_2s) - \alpha_{20}xs + a_{23}c - a_{24}s \\ w &= a_9x^3y^3 + a_{10}x^3y^2 + a_{11}x^3y + a_{12}x^3 + a_{13}x^2y^3 + a_{14}x^2y^2 + a_{15}x^2y + a_{16}x^2 + \\ &+ a_{17}xy^3 + a_{18}xy^2 + a_{19}xy + a_{20}xc + a_{21}y^3 + a_{22}y^2 + a_{23}s + a_{24}c + a_6xs + a_8(\psi_1s - \psi_2c) \end{aligned} \tag{2.1}$$

or in matrix form

$$\tilde{\mathbf{u}} = \mathbf{Pa}; \quad \tilde{\mathbf{u}} = \mathrm{col}\{u, v, w\}, \quad \mathbf{a} = \mathrm{col}\{a_1, \ldots, a_{24}\} \tag{2.2}$$

where $\tilde{\mathbf{u}}$ is the displacement vector for points of the finite element's middle surface, $\mathbf{a}$ is the vector of the unknown coefficients a_i of the polynomials in relations (2.1), and $\mathbf{P}$ is a connection matrix of order 3×24, the elements of which are multipliers of the coefficients a_i in relations (2.1).

Expressing the coefficients a_i in terms of nodal unknowns, we obtain

$$\begin{aligned} \mathbf{a} &= \mathbf{B}^{-1}\bar{\mathbf{u}} \\ \bar{\mathbf{u}} &= \mathrm{col}\{u_i, v_i, w_i, \vartheta_{1i}, \vartheta_{2i}, w_{xyi}, u_j, v_j, w_j, \vartheta_{1j}, \vartheta_{2j}, w_{xyj}, u_k, \ldots, w_{xyk}, u_n, \ldots, w_{xyn}\} \end{aligned} \tag{2.3}$$

where $\bar{\mathbf{u}}$ is the vector of the nodal displacements, angles of rotation and mixed derivatives of the deflection, and $\mathbf{B}$ is a matrix of order 24×24, the non-zero elements of which have the form

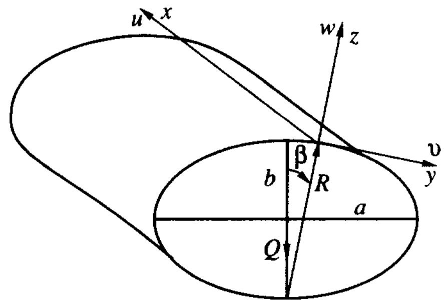

Fig. 1

$$
\begin{aligned}
&b_{1j}=p_{1j},\quad b_{2j}=p_{2j},\quad b_{3j}=p_{3j},\quad b_{4j}=(p_{3j})_x,\quad b_{5j}=(p_{2j}-(p_{3j})_y)/R,\\
&b_{6j}=(p_{3j})_{xy}\quad (x=-a_1, y=-b_1),\quad b_{7j}=p_{1j},\quad b_{8j}=p_{2j},\quad b_{9j}=p_{3j},\\
&b_{10j}=(p_{3j})_x,\quad b_{11j}=(p_{2j}-(p_{3j})_y)/R,\quad b_{12j}=(p_{3j})_{x\beta}\quad (x=-a_1, y=b_1),\\
&b_{13j}=p_{1j},\quad b_{14j}=p_{2j},\quad b_{15j}=p_{3j},\quad b_{16j}=(p_{3j})_x,\quad b_{17j}=(p_{2j}-(p_{3j})_\beta)/R,\\
&b_{18j}=(p_{3j})_{xy}\quad (x=a_1, y=-b_1),\quad b_{19j}=p_{1j},\quad b_{20j}=p_{2j},\quad b_{21j}=p_{3j},\\
&b_{22j}=(p_{3j})_x,\quad b_{23j}=(p_{2j}-(p_{3j})_y)/R,\quad b_{24j}=(p_{3j})_{xy}\quad (x=a_1, y=b_1),\\
&j=1,\dots,24,\quad a_1=L/(2m),\quad b_1=l/(2n)
\end{aligned}
\tag{2.4}
$$

were L and l are lengths of the shell's generatrix and direction respectively.

Substituting expression (2.3) into (2.2), we obtain the dependence of the displacements of the element's points on the nodal unknowns

$$\tilde{\mathbf{u}} = \mathbf{P}\mathbf{B}^{-1}\bar{\mathbf{u}} \tag{2.5}$$

There are six unknowns in each node, so every finite element has 24 degrees of freedom. We determine the nodal unknowns using Lagrange's variational equation $\delta\Pi = 0$, where Π is the total potential energy of the shell. The expression for the potential energy is written using the non-linear strain-displacement relations [1]. The equation $\delta\Pi = 0$ leads to a system of non-linear algebraic equations for the nodal unknowns. This system is solved by a step load method using the Newton–Kantorovich linearization method at each step, the equation of which can be written for the finite element in the form [2]

$$\mathbf{H}(\bar{\mathbf{u}}^n)\delta\bar{\mathbf{u}} = \mathbf{q}_e - \mathbf{G}(\bar{\mathbf{u}}^n),\quad \bar{\mathbf{u}}^{n+1} = \bar{\mathbf{u}}^n + \delta\bar{\mathbf{u}} \tag{2.6}$$

Here $\mathbf{H}$ is the Hessian matrix of the finite element, which is determined from the second variation of the strain potential energy, $\mathbf{q}_e$ is the vector of the nodal load and $\mathbf{G}$ is the potential energy gradient.

Equations of the type (2.6) for the whole shell are constructed [3] in the usual manner taking the boundary conditions into account. The boundary conditions are imposed in the following form: for the zero-valued nodal boundary displacements, their corresponding row of the Hessian matrix $\mathbf{H}$ and the corresponding component of the nodal load vector are set equal to zero, and a large number is substituted in place of the diagonal coefficient in the matrix $\mathbf{H}$.

We will seek a solution of the system of linear algebraic equations (2.6) by Crout's method using the factorization $\mathbf{H} = \mathbf{L}^T\mathbf{D}\mathbf{L}$ of the Hessian matrix into a diagonal matrix and two triangular matrices. The nodal displacements obtained are used to calculate the stresses and strains by well-known formulae. The stability is monitored by checking the positive definiteness of the Hessian matrix, which reduces to checking the positiveness of the elements of the diagonal matrix $\mathbf{D}$. The occurrence of negative elements corresponds to the shell's loss of stability.

After finding the value of the loading parameter, for which the equilibrium state is unstable, the form of the shell's loss of stability is sought from the solution of the system $\mathbf{H}\boldsymbol{\delta} = 0$, where $\boldsymbol{\delta}$ is the vector of bifurcational nodal displacements. For this purpose, we find one linearly dependent (degenerate) row of the matrix $\mathbf{H}$, corresponding to the first negative element of the matrix $\mathbf{D}$. The elements of this row and of the corresponding column of the matrix $\mathbf{H}$ are set equal to zero. A unity is put in place of the diagonal coefficient and the corresponding column, multiplied by the precritical displacement,

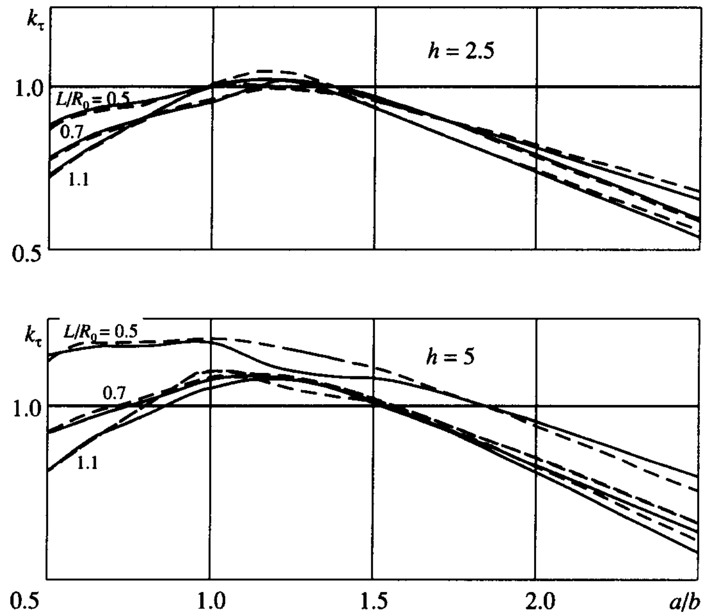

Fig. 2

corresponding to the degenerate row, is transferred to the right-hand side of the system. From the solution of the system thus obtained, the form of the shell's loss of stability is sought. The algorithm described is implemented by a computer program.

3. NON-LINEAR DEFORMATION AND STABILITY OF CYLINDRICAL SHELLS OF ELLIPTIC CROSS-SECTION FOR TRANSVERSE BENDING

Consider the problem of the non-linear deformation and stability of a cantilever ($u = \upsilon = w = w_x$) cylindrical shell under a transverse force Q, applied at the free edge. In this case

$$\frac{z^2}{b^2} + \frac{y^2}{a^2} = 1, \quad R = \frac{a^2 b^2}{d^3}, \quad d^2 = a^2 s^2 + b^2 c^2, \quad \psi_1 = -\frac{b^2 c}{d}, \quad \psi_2 = -\frac{a^2 s}{d}$$

The shell is strengthened by a rigid frame on the loaded side. The shell has a length $L = 300$–1100 mm, a thickness $h = 2.5$–5 mm and an equiperimeter radius of cross-section $R_0 = 1000$ mm (the equiperimeter radius of an ellipse is defined as the radius of the circumference that has the same perimeter as the ellipse). The shell is made of a material with a modulus of elasticiting $E = 0.7 \times 10^5$ MPa and a Poisson's ratio of 0.3.

Numerical investigations were made under assumption that the deformation was symmetric about the plane of longitudinal cross-section of the shell. In this case the shells were cut along the minor axis, and the symmetry boundary conditions ($u = w_y = w_{xy} = 0$) were imposed along the section line. Half of the shell was divided into 30 finite elements along the length and into 60 finite elements along the circumference.

For $h = 2.5$ and 5 mm, Fig. 2 shows graphs of the parameter $k_\tau = Q^*/Q_0$ against the ellipticity parameter a/b for linear (the dashed curves) and nonlinear (the solid curves) initial stress-strain states and for different lengths of the shell (Q^* is the critical value of the transverse force, $Q_0 = \pi R_0 S_b$, $S_b = 0.78CEh(h/R_0)^{5/4}(R_0/L)^{1/2}$ is the upper critical shear force for the twisting of a circular cylindrical shell with radius R_0, and $C = 0.93$ is an empirical coefficient). When the ellipticity parameter increases, the values of k_r first increase, reach a maximum, and then decrease; this is explained by the increase in the curvature in the region of the largest shear forces. The influence of the non-linearity of the initial state is small over practically the whole range of variation of a/b for all lengths and thicknesses of the shells.

Figure 3 shows graphs of the parameter k_τ against the shell's length for different values of the parameter a/b.

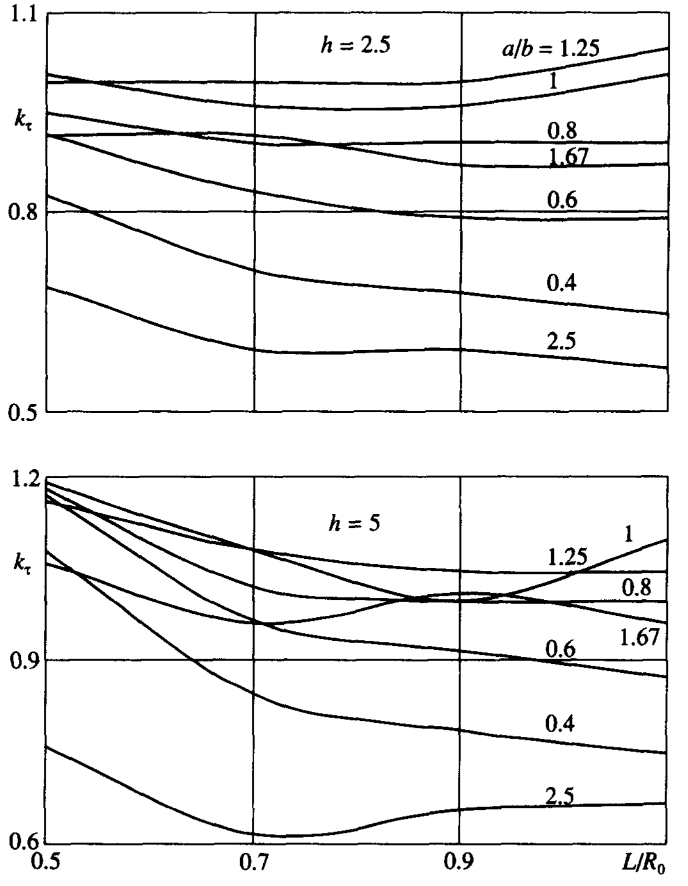

Fig. 3

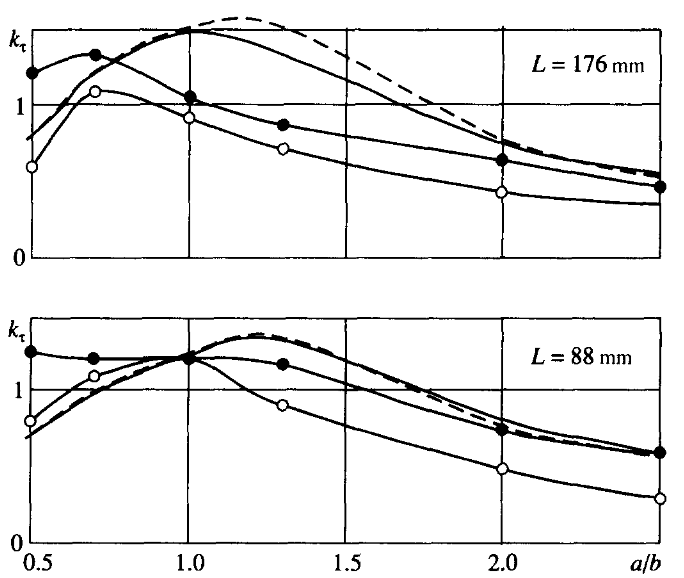

Fig. 4

Figure 4 shows graphs of the parameter k_τ against the parameter a/b for $R_0 = 44$ mm and $h = 0.05$ mm and experimental results [4] (the light circles correspond to the shell's loss of stability and the dark circles correspond to the loss of the shell's bearing strength). It can be seen that the values of the critical loads for elliptical shells are lower than for circular shells. For elliptical shells with $a/b < 1$, the results of

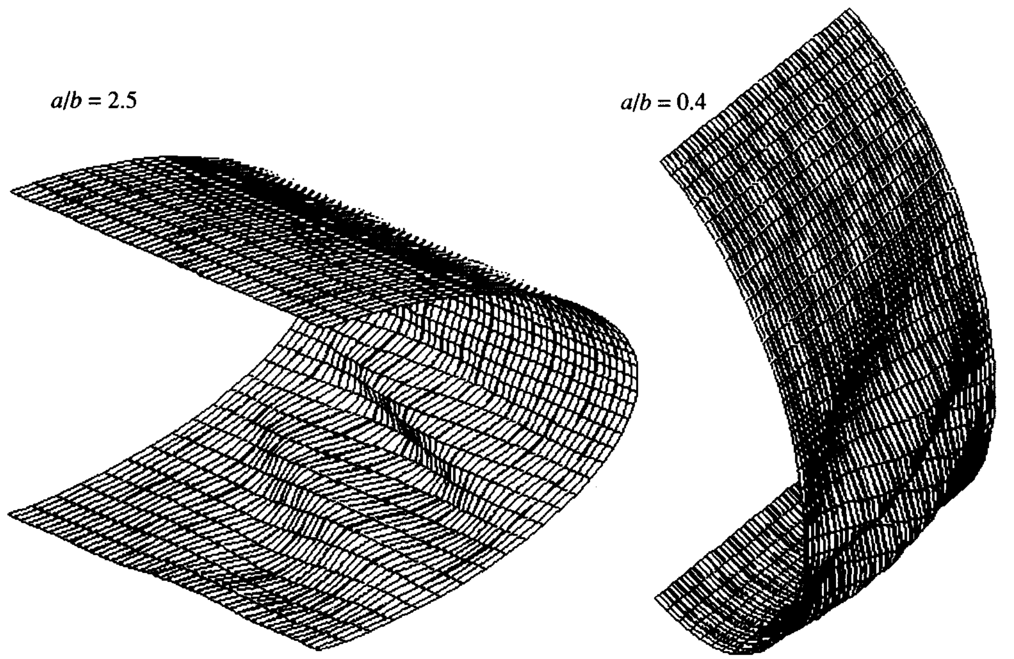

Fig. 5

Table 1

	$m \times n$							
a/b	10×13	15×20	20×27	25×33	10×40	20×40	25×40	30×40
	$L/R_0 = 0.5$				$L/R_0 = 1$			
0.4	1.48	1.28	1.125	1.125	0.545	0.77	0.73	0.73
1	1.24	1.13	1.18	1.18	1.19	1.19	1.08	1.08
2.5	0.87	0.81	0.79	0.79				

calculations are closer to the experimental values (shown by the light circles) and for elliptical shells with $a/b > 1$, the results of the calculations are closer to the experimental values (shown by the dark circles).

The forms of the loss of stability for shells with $L = 1100$ mm, $h = 2.5$ mm, $a/b = 2.5$ mm and $a/b = 0.4$ mm are shown in Fig. 5. It can be seen that high shells, like circular ones, loose stability mainly doe to the effect of the shear forces, with the formation of three inclined on the side surface, while shallow shells loose stability due to the simultaneous effect of compressive axial forces and shear forces, with the formation of one to three inclined folds on the lower part of the shell. This probably explains the above-mentioned agreement between the calculated and experimental values of the critical load.

The convergence of the solution as the number of finite elements increases, for shells with $R_0/h = 200$ and $L/R_0 = 0.5$ and 1.0 is shown in Table 1.

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